Dr. S. Chakraborty

INTEGRATING FACTORS: (I.F.)

Definition: An I.F. is a multiplying factor by which the D.E. $M(x,y)\,dx + N(x,y)\,dy = 0$ can be made exact.
i.e., if $\mu(x,y)$ is an I.F. of $M\,dx + N\,dy = 0$, then $\exists\ u(x,y)$ s.t. $\mu(M\,dx + N\,dy) = du$.

Now the question arises —
(i) whether or not I.F.s exist.
(ii) if an I.F. exists, is it unique?

Existence of I.F.s.

Theorem → If the equation $M\,dx + N\,dy = 0$ has one and only one solution, then there exists an infinity of I.F.s.

Proof → Let $f(x,y) = c$ [c is the arbitrary constant] be the g.s. of the D.E. $M\,dx + N\,dy = 0 \longrightarrow (1)$

Taking total differential of $f(x,y) = c$, we get

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = dc = 0 \longrightarrow (2).$$

Comparing (1) & (2), we get

$$\frac{\partial f}{\partial x}\Big/ M = \frac{\partial f}{\partial y}\Big/ N = \mu(x,y), \text{ say}.$$

$$\Rightarrow \frac{\partial f}{\partial x} = \mu M \quad \text{and} \quad \frac{\partial f}{\partial y} = \mu N$$

$\therefore$ From (2), we have

$$\mu(M\,dx + N\,dy) = df$$

$\Rightarrow$ an I.F. μ exists.

Let $\phi(f)$ be any function of f. Then

$$\mu\phi(f)(M\,dx + N\,dy) = \phi(f)\,df. \quad [\text{R.H.S. is exact}]$$

$\therefore$ $\mu\phi(f)$ is also an I.F.

Since $\phi(f)$ is an arbitrary function of f, there exists an infinity of I.F.s.

Finding an I.F.

1. An I.F. can be found by inspection.
2. There exist several rules for finding an I.F.

Examples for finding an I.F. by inspection.

① $x\,dy - y\,dx = 0$, I.F. $= \frac{1}{x^2}$; $\frac{1}{x^2}(x\,dy - y\,dx) = d\left(\frac{y}{x}\right)$.

$x\,dy - y\,dx = 0$, I.F. $= \frac{1}{x^2+y^2}$; $\frac{(x\,dy - y\,dx)/x^2}{(x^2+y^2)/x^2} = d\left\{\tan^{-1}\frac{y}{x}\right\}$.

$x\,dy - y\,dx = 0$, I.F. $= \frac{1}{xy}$; $\frac{dy}{y} - \frac{dx}{x} = d\left\{\log\frac{y}{x}\right\}$.

② $x\,dx + y\,dy = 0$; I.F. $= \frac{1}{x^2+y^2}$; $\frac{\frac{1}{2}d(x^2+y^2)}{x^2+y^2} = \frac{1}{2}d\left\{\log(x^2+y^2)\right\}$.

③ $y\,dx - x\,dy = 0$; I.F. $= \frac{1}{y^2}$; $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$.

- Solve the D.E.s. by rearranging the terms, or, by finding an I.F. by inspection:

① $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

$\Rightarrow (y\,dx + x\,dy) + xy\,(y\,dx - x\,dy) = 0$

Multiplying by the I.F. $\frac{1}{(xy)^2}$, we get

$$\frac{y\,dx + x\,dy}{(xy)^2} + \frac{y\,dx - x\,dy}{xy} = 0$$

a, $$\frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0. \Rightarrow \text{Integrating,}$$

$$-\frac{1}{xy} + \log x - \log y = \log c, \text{ [c is an arbitrary Constant]}$$

a, $$\log\left(\frac{x}{cy}\right) = \frac{1}{xy} \Rightarrow \boxed{x = cy\,e^{1/xy}} \leftarrow \text{g.s.}$$

② $x\,dy = y\,(y\log x - 1)\,dx$

By rearranging the terms,

$$x\,dy + y\,dx = y^2\log x\,dx$$

Multiplying by the I.F. $\frac{1}{(xy)^2}$, we get

$$\frac{x\,dy + y\,dx}{(xy)^2} = \frac{y^2\log x}{(xy)^2}\,dx \Rightarrow \frac{d(xy)}{(xy)^2} = \frac{\log x}{x^2}\,dx$$

Integrating $\Rightarrow -\frac{1}{xy} = \int z e^{-z}\,dz + c$, where $\log x = z$

or, $-\frac{1}{xy} = -ze^{-z} + \int 1.e^{-z}dz + c = -\frac{1}{x}\log x - \frac{1}{x} + c$

$\Rightarrow -\frac{1}{xy} = -\frac{1}{x}(\log x + 1) + c$

$\Rightarrow \underline{1 + cxy = y(1+\log x)}$ $\leftarrow$ g.s.

③ $\left\{x + y\cos\left(\frac{y}{x}\right)\right\}dx = x\cos\left(\frac{y}{x}\right)dy$

or, $x\,dx + (y\,dx - x\,dy)\cos\left(\frac{y}{x}\right) = 0$.

Dividing by x^2 or multiplying by the I. F. $\frac{1}{x^2}$, we get

$\frac{dx}{x} - \frac{(x\,dy - y\,dx)}{x^2}\cdot\cos\left(\frac{y}{x}\right) = 0$

or, $\frac{dx}{x} - d\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) = 0 \xrightarrow{\text{Integrating}}$

or, $\log x - \int\cos\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) = c$

or, $\underline{\log x - \sin\left(\frac{y}{x}\right) = c}$ $\leftarrow$ g.s.

④ $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$.

or, $\frac{x\,dy - y\,dx}{\sqrt{x^2+y^2}} = x\,dx$

or, $\frac{(x\,dy - y\,dx)/x^2}{\sqrt{x^2+y^2}/x^2} = x\,dx$.

or, $\frac{d\left(\frac{y}{x}\right)}{\sqrt{(x^2+y^2)/x^2}\cdot\frac{1}{x}} = x\,dx \Rightarrow \frac{d(y/x)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = dx$

Integrating $\Rightarrow \log\left\{\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right\} = x + c$.

or, $\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = e^{(x+c)}$

or, $\underline{y + \sqrt{x^2+y^2} = xe^{(x+c)}}$. $\leftarrow$ g.s.

⑤ Show that $\frac{1}{x^2}$ is an I. F. of $x\,dy - y\,dx = 0 \longrightarrow$ ①

<u>Solution</u>: Multiplying ① by $\frac{1}{x^2}$, we get $\left(-\frac{y}{x^2}\right)dx + \left(\frac{1}{x}\right)dy = 0$. $\to$ ②

Now Eq. ② is of the form $M\,dx + N\,dy = 0$,

where $M = -\frac{y}{x^2}$ and $N = \frac{1}{x} \Rightarrow \frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}$

$\therefore$ After multiplication of ① by $\frac{1}{x^2}$, ② becomes an Exact D.E.

And $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$. $\therefore \frac{1}{x^2}$ is an I. F.

RULE FOR FINDING I.F.

Rule-1.

(i) If $M\,dx + N\,dy = 0$ is both homogeneous and exact, then its complete primitive is $M.x + N.y = c$, provided the degree of homogeneity $\neq -1$.

(ii) If $M\,dx + N\,dy = 0$ is homogeneous but not exact, then its I.F. $= \dfrac{1}{Mx+Ny}$, provided $Mx+Ny \neq 0$.

Examples of Rule-1.

① Solve: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$. ——→ ①

Rule-1(ii). Here $M = x^2y - 2xy^2 = x^3\left(\frac{y}{x} - 2.\frac{y^2}{x^2}\right) \equiv x^3\phi\left(\frac{y}{x}\right)$

and $N = -x^3 + 3x^2y = x^3\left(-1 + 3.\frac{y}{x}\right) \equiv x^3\psi\left(\frac{y}{x}\right)$.

Both M and N are homogeneous functions of degree 3.

∴ The given D.E. is homogeneous.

To check the condition for exactness;

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy.$$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the D.E. is not exact.

$$\therefore \; Mx + Ny = (x^2y - 2xy^2).x + (-x^3 + 3x^2y).y$$
$$= \cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2 = x^2y^2 (\neq 0)$$

$$\therefore \; \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}, \quad (x^2y^2 \neq 0).$$

Multiplying ① by the I.F. $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0.$$

or, $\frac{1}{y}\,dx - \frac{x}{y^2}\,dy - \frac{2}{x}\,dx + \frac{3}{y}\,dy = 0$

or, $d\left(\frac{x}{y}\right) - d(2\log x) + d(3\log y) = 0$

Integrating $\Rightarrow \frac{x}{y} + \log\frac{y^3}{x^2} = \log c$, [c is constant]

$\Rightarrow \log\frac{cx^2}{y^3} = \frac{x}{y} \Rightarrow cx^2 = y^3 e^{x/y}$.

G.S.

② Solve: $3x^2y\,dx + (x^3+y^3)\,dy = 0$

Rule-1(i)

Here $M = 3x^2y = x^3\left(3\frac{y}{x}\right) \equiv x^3\phi\left(\frac{y}{x}\right)$

and $N = x^3+y^3 = x^3\left(1+\frac{y^3}{x^3}\right) \equiv x^3\psi\left(\frac{y}{x}\right)$.

Both M and N are homogeneous functions degree 3.

Also, $\frac{\partial M}{\partial y} = 3x^2 = \frac{\partial N}{\partial x} \Rightarrow$ D.E. is exact.

$\therefore$ The given D.E. is homogeneous (degree $\neq -1$) and exact.

$\therefore$ The complete primitive / g.s. is given by

$Mx + Ny = c$, c being an arbitrary constant.

or, $3x^3y + x^3y + y^4 = c \Rightarrow 4x^3y + y^4 = c$.

Do yourself: Solve the following D.Es

③ $(x^4+y^4)\,dx - xy^3\,dy = 0$ [Ans: $y^4 = 4x^4\log x + cx^4$]

④ $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$ [Ans: $cy = e^{y/x}$]

⑤ $y^2dx + (x^2-xy-y^2)\,dy = 0$ [Ans: $(x-y)y^2 = c(x+y)$]

Rule-2.

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, say, then I.F. $= e^{\int f(x)\,dx}$

of the D.E. $M\,dx + N\,dy = 0$.

Examples of Rule-2.

A linear first order D.E. is of the form:

$$\frac{dy}{dx} + P(x).y = Q(x) \longrightarrow ①.$$

$\Rightarrow \{P(x).y - Q(x)\}\,dx + 1.\,dy = 0$ [$M\,dx+N\,dy=0$ form]

Here $M = P(x).y - Q(x)$, $N = 1$.

$\frac{\partial M}{\partial y} = P(x)$, $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ D.E. is not exact.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x)$, which is a function of x only.

$\therefore$ By Rule-2, the I.F. $= e^{\int P(x)\,dx}$.

① Rule-2. Solve the D.E. $(x^2+y^2+1)\,dx - 2xy\,dy = 0 \longrightarrow ①$

Here $M = x^2+y^2+1$, $N = -2xy$.

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -2y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ D.E. is not exact.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}(2y+2y) = -\frac{2}{x} \equiv f(x)$, say.

$\therefore$ I.F. $= e^{\int(-\frac{2}{x})dx} = e^{-2\log x} = \frac{1}{x^2}$.

Multiplying ① by the I.F., we get.

$\frac{1}{x^2}(x^2+y^2+1)\,dx - \frac{1}{x^2}\cdot 2xy\,dy = 0$

$\Rightarrow \left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - 2\frac{y}{x}\,dy = 0$

$\Rightarrow \quad dx + \frac{1}{x^2}dx + \frac{y^2}{x^2}dx - \frac{2y}{x}dy = 0$

$\Rightarrow \quad dx + d\left(-\frac{1}{x}\right) + d\left(-\frac{y^2}{x}\right) = 0$

Integration $\Rightarrow \; x - \frac{1}{x} - \frac{y^2}{x} = c$

a, $\underline{x^2 - y^2 - 1 = c.x}$. G.S.

② Rule-2 Solve the D.E. $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0 \longrightarrow ①$

Here $M = xy^2 - e^{1/x^3}$, $N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ D.E. is not exact.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2y}(2xy+2xy) = \frac{4xy}{-x^2y} = -\frac{4}{x} \equiv f(x)$, say,

I.F. $= e^{\int f(x)dx} = e^{-\int \frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$.

Multiplying ① by the I.F., we get

$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}dy = 0 \longrightarrow ②$

Now this D.E. ② becomes an exact D.E, since

$\frac{\partial}{\partial y}\left\{\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right\} = \frac{2y}{x^3} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\right)$.

② can be expressed in the form $du = 0$,

a, $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \longrightarrow ③$, comparing

with ②, we get $\underbrace{\frac{\partial u}{\partial x} = \frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}}_{④}$; $\underbrace{\frac{\partial u}{\partial y} = -\frac{y}{x^2}}_{⑤}$

From ⑤, $\frac{\partial u}{\partial y} = -\frac{y}{x^2}$, we integrate partially w.r.t. y keeping x as constant,

$$u = -\int_{(x \text{ constant})} \frac{y}{x^2}\,dy + g(x),$$ $g(x)$ being an arbitrary function of x alone.

$$u = -\frac{y^2}{2x^2} + g(x)$$

Differentiating partially w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} = \cancel{\frac{y^2}{x^3}} + g'(x) = \cancel{\frac{y^2}{x^3}} - \frac{1}{x^4} e^{1/x^3} \quad [\text{from ④}]$$

$$\Rightarrow \quad g'(x) = -\frac{1}{x^4} e^{1/x^3}.$$

Integrating w.r.t. x, we get

$$\int g'(x)\,dx = -\int \frac{1}{x^4} e^{1/x^3}\,dx$$

$$\Rightarrow \quad g(x) = \frac{1}{3}\int e^{z}\,dz + c_1 \qquad \left[\text{put } \frac{1}{x^3} = z,\ -\frac{3}{x^4}\,dx = dz\right]$$

$$g(x) = \frac{1}{3} e^{1/x^3} + c_1,$$ c_1 is an arbitrary constant

∴ The g.s. of D.E. ① is given by

$$u(x, y) = c_2$$

or, $$-\frac{y^2}{2x^2} + \frac{1}{3} e^{1/x^3} + c_1 = c_2$$

or, $$3y^2 - 2x^2 e^{1/x^3} = c x^2 \quad [c = 6(c_1 - c_2)].$$

Do yourself:

③ Solve: $(x^2 + y^2 + x)\,dx + xy\,dy = 0$ [Ans: $3x^4 + 4x^3 + 6x^2y^2 = c$].

④ Solve: $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$ [Ans: $e^{x}(x^2 + y^2) = c$]